Theory of Equation
(Reciprocal Equation)

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Definition

An equation is said to be a Reciprocal equation if the reciprocal of any root of the equation be also the root of the same equation.

Theorem

A reciprocal equation remains unaltered by changing x by $\frac{1}{x}$. Therefore a necessary condition for $f(x) = 0$ be a reciprocal equation is that zero is not a root of it i.e. $f(0) \neq 0$.

Proof: Let $a_0x^n + a_1x^{n-1} + \ldots + a_n = 0 \; (a_n \neq 0) \ldots\ldots\ldots\ldots$ (i) be an equation. Then the equation whose roots are the reciprocals of the roots of the given equation is

$$\left(\frac{1}{x}\right)^n + p_1\left(\frac{1}{x}\right)^{n-1} + p_2\left(\frac{1}{x}\right)^{n-2} + \ldots\ldots + p_{n-1}\left(\frac{1}{x}\right) + p_n = 0$$

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i.e. $a_n x^n + a_{n-1} x^{n-1} + \dots\dots\dots\dots + a_0 = 0 \dots\dots\dots\dots\dots$...(ii)
If (i) is a reciprocal equation then (i) and (ii) are identical.
Comparing the coefficients we get
$a_0 = ka_n, a_1 = ka_{n-1}, \dots\dots\dots\dots, a_n = ka_0$ where, k is a real number.

Now, $a_0 = ka_n = k.k.a_0 = k^2 a_0 \therefore k^2 = 1 \Rightarrow k = \pm 1$

Case -1: If$k = 1$, $a_0 = a_n, a_1 = a_{n-1}, \dots\dots\dots\dots, a_n = a_0$. In this case the equation is said to be a reciprocal equation of *first type* or *first class* or *first kind.*

Case-2: If$k = -1$, $a_0 = -a_n, a_1 = -a_{n-1}, \dots\dots\dots\dots, a_n = -a_0$. In this case the equation is said to be a reciprocal equation of second *type* or *second class* or *second kind.*

Definition

Standard form of reciprocal equation: A reciprocal equation is said to be of the *standard from* if it is of the first type and even degree.

Theorem

If $f(x) = 0$ be a reciprocal equation of degree n and of the firs type then $f(x) = x^n f\left(\frac{1}{x}\right)$.
Conversely, if $f(x) = 0$ be a polynomial of degree n and $f(x) = x^n f\left(\frac{1}{x}\right)$ then $f(x) = 0$ is a reciprocal equation of first type

Proof: Let,
$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots\ldots\ldots + a_n \, (a_0 \neq 0, a_n \neq 0)$
Since $f(x)$ is a reciprocal equation of first type then
$a_0 = a_n, a_1 = a_{n-1}, a_2 = a_{n-2}, \ldots\ldots\ldots\ldots$

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Since $f(x)$ is a reciprocal equation of first type then
$a_0 = a_n, a_1 = a_{n-1}, a_2 = a_{n-2}, ...$

Therefore, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots\ldots\ldots + a_0$

$$= x^n \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \ldots\ldots\ldots\ldots + a_n \right) = x^n f\left(\frac{1}{x}\right)$$

Conversely,

let $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots\ldots\ldots + a_n \, (a_0 \neq 0, a_n \neq 0)$.

Therefore, $x^n f\left(\frac{1}{x}\right) = a_0 + a_1 x + \ldots\ldots\ldots + a_n x^n$

Since, $f(x) = x^n f\left(\frac{1}{x}\right) \therefore a_0 = a_n, a_1 = a_{n-1}, a_2 = a_{n-2}, \ldots\ldots\ldots\ldots$

This shows that $f(x) = 0$ is a reciprocal equation of first type.

Theorem

If $f(x) = 0$ *be a reciprocal equation of degree n and 2nd type iff* $(x) = -x^n f\left(\frac{1}{x}\right)$.
Conversely, if $f(x) = -x^n f\left(\frac{1}{x}\right)$ *then* $f(x) = 0$ *is a reciprocal equation of 2nd type.*

Proof: Let,
$f(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots\ldots\ldots + a_n \,(a_0 \neq 0, a_n \neq 0)$
Since $f(x)$ is a reciprocal equation of second type then
$a_0 = -a_n, a_1 = -a_{n-1}, a_2 = -a_{n-2}, \ldots\ldots\ldots\ldots$
Therefore, $f(x) = -a_nx^n - a_{n-1}x^{n-1} - \ldots\ldots\ldots\ldots - a_0$

$$= -x^n\left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \ldots\ldots\ldots\ldots + a_n\right) = -x^n f\left(\frac{1}{x}\right)$$

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Therefore, $-x^n f\left(\frac{1}{x}\right) = -a_0 - a_1 x - .... - a_n x^n$
Since, $f(x) = -x^n f\left(\frac{1}{x}\right)$
$\therefore a_0 = -a_n, a_1 = -a_{n-1}, a_2 = -a_{n-2}, ...$
This shows that $f(x) = 0$ is a reciprocal equation of second type.

Theorem

A reciprocal equation of standard form can always be depressed to another of half of dimension.

Proof: Let the equation is$a_0x^{2m} + a_1x^{2m-1} + a_2x^{2m-2} + ... + a_{2m} = 0$.
Since it in standard form, $a_0 = a_{2m}$, $a_1 = a_{2m-1}, \cdots,$
$a_{m+1} = a_{m-1}$.
Therefore the equation can be written as
$a_0\left(x^{2m} + 1\right) + a_1x\left(x^{2m-1} + 1\right) + ... + a_mx^m = 0$
$\Rightarrow a_0\left(x^m + \frac{1}{x^m}\right) + a_1\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + ... + a_m = 0$ .... (i)

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Let, $x + \frac{1}{x} = t$ and$x^r + \frac{1}{x^r} = u_r$.
Now, $\left(x + \frac{1}{x}\right)\left(x^{r-1} + \frac{1}{x^{r-1}}\right) = \left(x^r + \frac{1}{x^r}\right) + \left(x^{r-2} + \frac{1}{x^{r-2}}\right)$

$$\Rightarrow u_r = tu_{r-1} - u_{r-2}$$

But, $u_0 = 2, u_1 = t$. Taking $r = 2, 3, 4, ...$we have
$u_2 = t^2 - 2, u_3 = t^3 - 3t, u_4 = t^4 - 4t^2 + 2 \ldots\ldots\ldots\ldots\ldots\ldots$ (ii)
Using (ii), (i) can be expressed as an equation in t of degree m.

Example

Solve the equation$6x^4 + 35x^3 + 62x^2 + 35x + 6 = 0$.

Sol. This is a reciprocal equation of even degree and first type. Thus it is in the standard form so it can be written as

$$6\left(x^4 + 1\right) + 35x\left(x^2 + 1\right) + 62x^2 = 0$$

Dividing both side by x^2 we get

$$6\left(x^2 + \frac{1}{x^2}\right) + 35\left(x + \frac{1}{x}\right) + 62 = 0$$

Putting $x + \frac{1}{x} = t$ we get

$$\therefore 6\left(t^2 - 2\right) + 35t + 62 = 0$$

Or, $6t^2 + 35t + 50 = 0$
Or, $(3t + 10)(2t + 5) = 0$

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Or, $6t^2 + 35t + 50 = 0$
Or, $(3t + 10)(2t + 5) = 0$

This gives, $t=-\frac{10}{3}, t=-\frac{5}{2}$
When, $t=-\frac{10}{3}$ then, $x+\frac{1}{x}=-\frac{10}{3}$
Or,$3x^2+10x+3=0$
Or, $(x+3)(3x+1)=0 \therefore x=-3,-\frac{1}{3}$
When, $t=-\frac{5}{2}$ then, $x+\frac{1}{x}=-\frac{5}{2}$
Or,$2x^2+5x+2=0$
Or, $(x+2)(2x+1)=0 \therefore x=-2,-\frac{1}{2}$

Example

Solve the equation $x^7 + 4x^6 + 4x^5 + x^4 - x^3 - 4x^2 - 4x - 1 = 0$.

Sol. This is reciprocal equation of odd degree and second type. It can be written as

$$(x^7 - 1) + 4\left(x^6 - x\right) + 4\left(x^5 - x^2\right) + \left(x^4 - x^3\right) = 0$$

This gives,

$$(x - 1)\left\{\left(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\right) + 4x\left(x^4 + x^3 + x^2 + x +\right.\right.$$

Thus either $x = 1$
Or, $x^6 + 5x^5 + 9x^4 + 10x^3 + 9x^2 + 5x + 1 = 0$

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It can be written as

$$\left(x^6+1\right)+5\left(x^5+x\right)+9\left(x^4+x^2\right)+10x^3=0$$

Dividing both side by x^3 we get

$$\left(x^3+\frac{1}{x^3}\right)+5\left(x^2+\frac{1}{x^2}\right)+9\left(x+\frac{1}{x}\right)+10=0$$

Putting $x+\frac{1}{x}=t$ we get

$$t^3-3t+5t^2-10+9t+10=0$$

Or, $t^3+5t^2+6t=0$
Or, $t\left(t^2+5t+6\right)=0$
Either, $t=0$ or, $t^2+5t+6=0$

Now, $t = 0 \Rightarrow x + \frac{1}{x} = 0$ This gives $x = \pm i$
Again, $t^2 + 5t + 6 = 0$ gives $(t+3)(t+2) = 0$ which gives$t = -3, -2$.
When, $t = -3$

$$x + \frac{1}{x} = -3 \Rightarrow x^2 + 3x + 1 = 0 \Rightarrow x = \frac{-3 \pm \sqrt{5}}{2}.$$

When, $t = -2$.

$$x + \frac{1}{x} = -2 \Rightarrow x^2 + 2x + 1 = 0 \Rightarrow x = -1, -1.$$

Thus the roots are$1, -1, -1, \pm i, \frac{-3 \pm \sqrt{5}}{2}$.


THANK YOU